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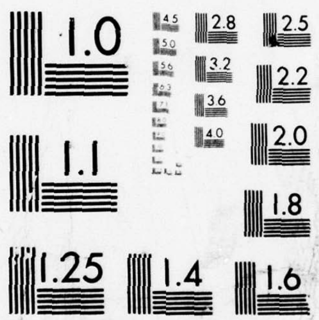
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REPEATED MEASUREMENTS DESIGNS, III

By

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July 1978

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Repeated measurements designs (RM designs) are experiments in which an experminetal unit is exposed to a sequence of treatments during a number of periods (days). A measurement taken on a unit depends on the treatment administered in the previous period (residual effect), the unit itself (unit effect) and the present period (period effect).

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20. Abstract continued

balanced and connected RM designs when the number of periods is less than the number of treatments. In the last section we discuss the structure and the ranks of the C-matrices for direct and residual effects which arise from these designs, as well as the form of the best linear unbiased estimators of estimable contrasts of residual and direct effects.

REPEATED MEASUREMENTS DESIGNS, III
BALANCED CONNECTED DESIGNS

By

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ABSTRACT

Repeated measurements designs (RM designs) are experiments in which an experimental unit is exposed to a sequence of treatments during a number of periods (days). A measurement taken on a unit depends on the treatment administered in the previous period (residual effect), the treatment given during the present period (direct effect), the unit itself (unit effect) and the present period (period effect).

In the first part of the paper, after defining the concept of balanced and connected RM design, we proceed in constructing large families of balanced and connected RM designs when the number of periods is less than the number of treatments. In the last section we discuss the structure and the ranks of the C-matrices for direct and residual effects which arise from these designs, as well as the form of the best linear unbiased estimators of estimable contrasts of residual and direct effects.

REPEATED MEASUREMENTS DESIGNS, III
BALANCED CONNECTED DESIGNS

By

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1. Introduction

A statistical design in which experimental units are used repeatedly by exposing them to a sequence of treatments is called a repeated measurements design (RM design). In a RM design we administer t treatments during p periods to n experimental units. The parameters t and p are usually known to the experimenter. RM designs have been considered in the literature under a variety of names such as crossover or changeover designs, time series designs or before-after designs. In these designs, apart from the direct treatment effects, residual or "carry-over" effects are usually assumed. It has been shown by A. Hedayat and K. Afsarinejad (1978), that if certain balance for the residual and direct effects is assumed, these designs become optimal in a large class of designs. The cyclic RM designs that we shall construct (precise definitions will follow), apart from having these balancing properties, are also connected, i.e., all contrasts of residual and direct effects are estimable. Their cyclic structure makes them easy to implement in practice. We also give the best linear unbiased estimators (b.l.u.e.'s) for some of the contrasts. Tables of such designs for small values of t and p can be found at the end of the paper. The authors hope that they would suffice for most practical applications.

2. Definitions and Model

Let us present the set-up in which we will be working. Suppose d is a two dimensional array. If a is in position $d(i-1,j)$ in d and b is in position $d(i,j)$ in d we say that a precedes b (in the order of application). We say that a follows b if b precedes a.

Definition 2.1. A balanced $RM(t,p,n,\lambda_1,\lambda_2)$ design is a $p \times n$ array d whose entries involve t treatments (labeled $0,1,\ldots,t-1$), satisfying the following conditions:

(a) the treatments along any column are distinct.

(b) A treatment occurs exactly λ_1 times in each row of d.

(c) In the order of application, each treatment is preceded in d by each other treatment exactly λ_2 times.

Part (a) in the above definition has been included mostly because of esthetic statistical reasons, such as treating a column in the array as a fixed size block, etc... We refer to the treatment in the (i,j)th entry of the design d by $d(i,j)$. The basic divisibility conditions for the existence of a balanced $RM(t,p,n,\lambda_1,\lambda_2)$ design are the following:

$$n = \lambda_1 t \quad \text{and} \quad \lambda_1(p-1) = \lambda_2(t-1).$$

Definition 2.2. A shift starting at i of $(0,1,2,\ldots,t-1)$ is $(i,i+1,\ldots,t-1,0,1,\ldots,i-1)$.

Definition 2.3. We say that two shifts of $(0,1,2,\ldots,t-1)$ are distinct if they start with different symbols.

Definition 2.4. A balanced $RM(t,p,n,\lambda_1,\lambda_2)$ design is called cyclic if it can be decomposed into λ_1 subarrays of size $p\times t$ and the rows of each $p\times t$ subarray are shifts of $(0,1,2,\ldots,t-1)$.

Throughout the paper the underlying model will be:

$$Y_{ij} = \mu + \alpha_i + \beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)} + e_{ij};\ 1 \leq i \leq p,\ 1 \leq j \leq n$$

where e_{ij} are assumed to have mean zero and covariance matrix $\sigma^2 I$. The overall effect is μ, α_i is the effect of the i^{th} period, β_j is the effect of the j^{th} experimental unit, $\tau_{d(i,j)}$ is the direct effect of treatment $d(i,j)$ and $\rho_{d(\kappa,\ell)}$ is the residual effect of treatment $d(\kappa,\ell)$. During the first period (i.e., for $i=1$), no residual effects are assumed, so $\rho_{d(0,j)}$ are zero for all j.

3. Construction of Balanced RM Designs with minimal number of units

In this section we construct families of cyclic balanced RM designs. The main result is contained in Theorem 3.1. We begin with a couple of useful propositions. As we shall see in the proposition that follows, the advantage of constructing balanced RM designs by shifts is that condition (c) of Definition 2.1 needs to be satisfied for one treatment only. Let the treatments be $\{0,1,2,\ldots,t-1\}$.

Proposition 3.1. *A* $p \times n$ *array* d, *with* λ_1 *subarrays of size* $p \times t$ *each, is a cyclic balanced* $RM(t,p,n,\lambda_1,\lambda_2)$ *design if all the rows in each* $p \times t$ *subarray are distinct shifts of* $(0,1,2,\ldots,t-1)$ *and, in* d, *the treatment* 0 *is followed in order of application by every other treatment exactly* λ_2 *times*.

Proof: Since the shifts in each $p \times t$ subarray are assumed distinct, (a) is satisfied. Clearly (b) holds. To prove (c), fix treatment j. By assumption, 0 precedes j exactly λ_2 times. So let i be different of 0 and j. The treatment $j-i \pmod t$ is nonzero and hence, preceded by 0 λ_2 times in d. There are therefore exactly λ_2 rows of length t in various $p \times t$ subarrays in which $j-i \pmod t$ is preceded by 0. In all these rows, j is preceded by i, because each such row is assumed to be a shift of $(0,1,2,\ldots,t-1)$. Therefore j is preceded by i exactly λ_2 times.

The following proposition reduces the construction of a cyclic balanced RM design to a partition problem and is our tool for constructing these designs.

Proposition 3.2. *Let $\lambda_1(p-1) = (t-1)$ (i.e., $\lambda_2 = 1$). A cyclic balanced $RM(t,p,n,\lambda_1,1)$ design d exists if and only if $\{1,2,3,\ldots,t-1\} = \bigcup_{i=1}^{\lambda_1} P_i$ (disjoint), $|P_i| = p-1$ such that for some ordering of the elements in P_i, the successive partial sums of these elements are all distinct and nonzero (modulo t), for all i, $1 \leq i \leq \lambda_1$.*

Proof: For simplicity of notation, in the proof that follows we will not use the $d(i,j)$ notation introduced earlier. Note that the condition on the successive partial sums in P_i to be distinct and nonzero (mod t) can be expressed equivalently, by subtracting two partial sums, as $\sum_{\ell=j}^{k} s_{i\ell} \not\equiv 0 \pmod t$ for all $1 \leq j < k \leq p-1$, where $P_i = \{(s_{ij})\}$.

Suppose we have a cyclic balanced $RM(t,p,n,\lambda_1,1)$ design d. Let f_{ij} map the entries of the j^{th} row into the corresponding entries of the $(j+1)^{th}$ row, in the i^{th} $p \times t$ subarray of d. Because d is a cyclic design, f_{ij} has the following property: $f_{ij}(x) = f_{ij}(0) + x$. Let $s_{ij} = f_{ij}(0)$. Define $P_i = \{(s_{ij})\}_{1\leq j\leq (p-1)} = \{(s_{i1}, s_{i2}, s_{i3}, \ldots, s_{i(p-1)})\}$.

$\bigcup_{i=1}^{\lambda_1} P_i = \{1,2,3,\ldots,t-1\}$ holds because 0 is preceded in d by everything else exactly once. On the other hand, in the i^{th} pxt subarray of d, the entries along any column are distinct, so in particular,

$f_{i1}(0), f_{i2}(f_{i1}(0)), \ldots, f_{i(p-1)}(f_{i(p-2)}(\ldots(f_{i1}(0))\ldots))$ are distinct. But $f_{ij}(f_{i(j-1)}(\ldots(f_{i1}(0))\ldots)) = s_{i1}+s_{i2}+\ldots+s_{ij}$.

So $\sum_{\ell=j}^{k} s_{i\ell} \not\equiv 0 \pmod t$ follows (by subtracting two such partial sums), for all $1 \leq j < k \leq (p-1)$.

Conversely, assume $P_i = \{(s_{i1}, s_{i2}, s_{i3}, \ldots, s_{i(p-1)})\}$. Construct the cyclic balanced $RM(t,p,n,\lambda_1,1)$ design d as follows: In the i^{th} pxt subarray let the zeros be followed by the s_{ij}'s in the same order as they appear in P_i. Then complete the subarray by shifts. Down a column (without loss the first column) we have $0, s_1, s_1+s_2, \ldots, s_1+s_2+\ldots+s_{(p-1)}$ which are all distinct by assumption. Since we complete the subarray by shifts, the entries along any column of the subarray will also be distinct. With the help of Proposition 3.1, we conclude the proof.

Remark. An analogous result to Proposition 3.2 has been established in which the restriction $\lambda_2 = 1$ is not imposed. The authors chose to include the present version because of its direct impact on the results that follow and because of its simplicity.

We introduce two examples to illustrate how a partition, as described in Proposition 3.2, can be associated with a cyclic balanced RM design.

Example 1. Let $t=9$, $p=3$, $n=36$, $\lambda_1=4$ and $\lambda_2=1$. Consider $P_1 = \{(1,7)\}$, $P_2 = \{(2,6)\}$, $P_3 = \{(3,5)\}$ and $P_4 = \{(4,8)\}$. These P_i's satisfy the requirements of the previous proposition. The associated cyclic balanced RM(9,3,36,4,1) design is:

$$\begin{array}{ccccccccc} 0&1&2&3&4&5&6&7&8\\ 1&2&3&4&5&6&7&8&0\\ 8&0&1&2&3&4&5&6&7 \end{array} \qquad \begin{array}{ccccccccc} 0&1&2&3&4&5&6&7&8\\ 2&3&4&5&6&7&8&0&1\\ 8&0&1&2&3&4&5&6&7 \end{array}$$

$$\begin{array}{ccccccccc} 0&1&2&3&4&5&6&7&8\\ 3&4&5&6&7&8&0&1&2\\ 8&0&1&2&3&4&5&6&7 \end{array} \qquad \begin{array}{ccccccccc} 0&1&2&3&4&5&6&7&8\\ 4&5&6&7&8&0&1&2&3\\ 3&4&5&6&7&8&0&1&2 \end{array} .$$

Example 2. For $t = 9$, $p = 5$, $\lambda_1 = 2$, let $P_1 = \{(1,7,3,5)\}$ and $P_2 = \{(8,2,6,4)\}$. This partition corresponds to the design:

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\ 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \qquad \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \end{array} .$$

Note, however, that the partition $P_1 = \{(1,7,3,4)\}$ and $P_2 = \{(2,6,4,8)\}$ does not generate a cyclic balanced RM design in the same fashion. This happens because, in P_2, $6 + 4 + 8 = 0 \pmod 9$.

In the next theorem we give a constructive proof of the existence of cyclic balanced $RM(t,p,n,\lambda_1,1)$ designs for all sets of parameters satisfying $p < t$ and the basic divisibility conditions. It will be shown later that all these designs are connected (i.e., all linear contrasts of direct and residual effects are estimable).

Theorem 3.1. The necessary and sufficient conditions for the existence of a cyclic balanced $RM(t,p,n,\lambda_1,1)$ *design with* $p < t$ *are* $n = \lambda_1 t$ *and* $\lambda_1(p-1) = (t-1)$.

Proof: (by construction).

We first give the partitions that we use to construct the designs, for all cases.

case 1. (p even). Consider the following partition of $\{1,2,3,\ldots,t-1\}$:

$$P_i = \{(i,(t-1)-i,\ i+\lambda_1,\ (t-1)-(i+\lambda_1),\ i+2\lambda_1,\ (t-1)-(i+2\lambda_1),\ldots$$

$$\ldots,\ i+j\lambda_1,\ (t-1)-(i+j\lambda_1),\ldots,i+\frac{(p-2)}{2}\lambda_1)\}\quad\text{for}$$

$0 \leq j < \frac{(p-2)}{2}$, $1 \leq i < \lambda_1$, and for $i = \lambda_1$

$$P_{\lambda_1} = \{(\lambda_1,\ (t-1)-\lambda_1,\ 2\lambda_1,\ (t-1)-2\lambda_1,\ 3\lambda_1,\ (t-1)-3\lambda_1,\ldots$$

$$\ldots,j\lambda_1,\ (t-1)-j\lambda_1,\ldots,(t-1))\}\quad\text{where}\quad 1 \leq j \leq \frac{(p-2)}{2}\ .$$

case 2.1. (p odd and $\lambda_1 > 2$). Define the partition as follows:

$$P_i = \{(i,(t-1)-i,\ i+\lambda_1,\ (t-1)-(i+\lambda_1),\ i+2\lambda_1,\ (t-1)-(i+2\lambda_1),\ldots$$

$$\ldots,\ i+j\lambda_1,\ (t-1)-(i+j\lambda_1),\ldots,i+\frac{(p-3)}{2}\lambda_1,\ (t-1)-(i+\frac{(p-3)}{2}\lambda_1))\}$$

for $0 \leq j \leq \frac{(p-3)}{2}$, $1 \leq i < \lambda_1$ and for $i = \lambda_1$

$$P_{\lambda_1} = \{(\lambda_1,\ (t-1)-\lambda_1,\ 2\lambda_1,\ (t-1)-2\lambda_1,\ 3\lambda_1,\ (t-1)-3\lambda_1,\ldots$$

$$\ldots,j\lambda_1,\ (t-1)-j\lambda_1,\ldots,\frac{(p-1)}{2}\lambda_1,\ (t-1))\}\quad\text{for}\quad 1 \leq j < \frac{(p-1)}{2}.$$

case 2.2. (p odd and $\lambda_1 = 2$).

Let $P_1 = \{(1,(t-1)-1,\ 3,\ (t-1)-3,\ 5,\ (t-1)-5,\ldots$

$$\ldots,\ 1+2j,\ (t-1)-(1+2j),\ldots,1+2\frac{(p-3)}{2},\ (t-1)-(1+2\frac{(p-3)}{2}))\}$$

for $0 \leq j \leq \frac{p-3}{2}$, and

$P_2 = \{(t-1), 2, (t-1)-2, 4, (t-1)-4, 5, (t-1)-6, \ldots$
$\ldots, 2j, (t-1-2j, \ldots, (p-1))\}$ for $1 \leq j \leq \frac{(p-3)}{2}$
and $(p-1) = \frac{(t-1)}{2}$.

The proof consists in showing that the conditions stated in Proposition 3.2 are satisfied for all the partitions above. In all these cases it is straightforward to check that

$\bigcup_{i=1}^{\lambda_1} P_i = \{1,2,3,\ldots,t-1\}$ (disjoint) and that $|P_i| = p-1$

for all $1 \leq i \leq \lambda_1$.

To check the condition $\sum_{\ell=j}^{k} s_{i\ell} \not\equiv 0 \pmod t$ for all $1 \leq j < k \leq (p-1)$, where $P_i = \{(s_{ij})\}_{1 \leq j \leq (p-1)}$ is the P_i defined above, we note that any two (suitable) neighboring entries in P_i always sum up to t-1, for all $1 \leq i \leq \lambda_1$. This fact enables us to show that an arbitrary sum $\sum_{\ell=j}^{k} s_{i\ell}$, as above, is never a multiple of t. A detailed verification for all cases is given in the Appendix.

In practice, the number of treatments, t, and (possibly) the number of periods, p, are known to the experimenter. It is usually of interest to construct a balanced RM design d, with known t and p, in which the number of units, n, is minimal. This can be achieved (as the divisibility conditions show), if and only if λ_1 and λ_2 are relatively prime. The designs in Theorem 3.1 have minimal n since $\lambda_2 = 1$.

It is easy to extend the validity of the previous theorem in the following sense:

Corollary 3.1. *The necessary and sufficient conditions for the existence of a cyclic balanced* $RM(t,p,n,\lambda_1,\lambda_2)$ *design with* $p < t$ *and* $\lambda_2 | \lambda_1$ *are* $n = \lambda_1 t$ *and* $\lambda_1(p-1) = \lambda_2(t-1)$.

Proof: Let $\lambda_1 = m\lambda_2$. Construct, as in the previous theorem, a cyclic $RM(t,p,n,m,1)$ design. Take λ_2 copies of this design to obtain a cyclic balanced $RM(t,p,n,\lambda_1,\lambda_2)$ design, as desired.

Even though shifts prove successful in constructing balanced RM designs with $p < t$, this is not the case for $p = t$. Note that for odd t, cyclic balanced $RM(t,t,t,1,1)$ designs do not exist. This is so because in this case we only have the trivial partition $P_1 = \{1,2,3,\ldots,t-1\}$. Clearly, $\sum_{j=1}^{t-1} j = \frac{(t-1)}{2}t = 0 \pmod t$, no matter how we order the elements in P_1. Nonexistence is now assured by Proposition 3.2.

The construction of balanced $RM(t,p,n,\lambda_1,\lambda_2)$ designs for $\lambda_2 \nmid \lambda_1$ is researched presently. Some results have been obtained and they will be presented in a subsequent paper.

4. Estimability and Estimation.

A statistical design is of practical use if at least the differences between various parameters of interest can be unbiasedly estimated from the observations. In this section we shall be mainly concerned with the concept of estimability, so we recall the usual definitions.

Let β be the vector of parameters and Y be the vector of observations. Suppose we assume m effects in our linear model. Then we can write:

$$\beta' = (\beta_1\beta_2\cdots\beta_m)'$$

where β_i is the vector of parameters associated with the i^{th} effect.

Definition 4.1. The difference between two parameters is called an elementary contrast. A linear combination of parameters $\sum_j c_j\gamma_j$ is called a (linear) contrast if $\sum_j c_j = 0$.

Definition 4.2. A linear function of parameters is estimable if it is equal to the expected value of some linear function of the observations; i.e., $\ell'\gamma$ is estimable if $\ell'\gamma = E(t'Y)$ for some vector t of constants.

Definition 4.3. A design is connected for the i^{th} effect if all the linear contrasts of parameters of the i^{th} effect are estimable. A design is called connected if it is connected for all the assumed effects.

We should mention that the subspace of linear contrasts of t parameters is of dimension $t-1$ and is spanned by the elementary contrasts.

The following lemma will be helpful in proving the next proposition.

Lemma 4.1. *Let $S = \{a_0-a_r, a_1-a_{r+1}, a_2-a_{r+2},\ldots,a_{t-1}-a_{r+(t-1)}\}$, where a_i are constants, and let $(r,t) = 1$. Then a_i-a_j $(i \neq j)$ can be expressed as a sum of elements of S. (All the indexing is modulo t).*

Proof: Since $(r,t) = 1$, the Z-module generated by r in Z_t equals Z_t so we can write $S = \{a_0-a_r, a_r-a_{2r}, a_{2r}-a_{3r},\ldots, a_{(t-1)r}-a_{tr}\}$. If $i = kr$ and $j = mr$, then $a_i - a_j =$
$= (a_{kr}-a_{(k+1)r}) + (a_{(k+1)r}-a_{(k+2)})+\ldots+(a_{(m-1)r}-a_{mr})$.

In order to show that the designs constructed in the previous section are connected, we introduce the following:

Proposition 4.1. *Assume $p < t$. A cyclic balanced $RM(t,p,n,\lambda_1,\lambda_2)$ design d is connected for direct and residual effects if there exist two $p\times t$ subarrays of d such that the zeros in the first period of these subarrays are followed by a and b respectively, and $(b-a,t) = 1$.*

<u>Proof</u>: The necessary conditions $n = \lambda_1 t$, $\lambda_1(p-1) = \lambda_2(t-1)$ together with $p < t$ imply $\lambda_1 > \lambda_2 \geq 1$. The design d has, therefore, at least two $p \times t$ subarrays. Suppose they are as follows:

$$\begin{pmatrix} 0 & 1 & 2 & \ldots\ldots & (t-1) \\ a & a+1 & a+2 & \ldots & a+(t-1) \\ \multicolumn{5}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & \ldots\ldots & (t-1) \\ b & b+1 & b+2 & \ldots & b+(t-1) \\ \multicolumn{5}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \end{pmatrix} \quad (*)$$

where a and b satisfy $(b-a,t) = 1$. Consider the entries $(1,j)$, $(2,j)$, $(1,t+j)$, $(2,t+j)$ in $(*)$. We have $E(Y_{1j}) = \mu + \alpha_1 + \beta_j + \tau_{j-1}$, $E(Y_{1,t+j}) = \mu + \alpha_1 + \beta_{t+j} + \tau_{j-1}$, $E(Y_{2j}) = \mu + \alpha_2 + \beta_j + \tau_{a+j-1} + \rho_{j-1}$ and $E(Y_{2,t+j}) =$ $= \mu + \alpha_2 + \beta_{t+j} + \tau_{b+j-1} + \rho_{j-1}$. So

$$E\left[\tfrac{1}{2}(Y_{1j}-Y_{2j} + Y_{2,t+j}-Y_{1,t+j})\right] = \tau_{b+j-1} - \tau_{a+j-1}, \text{ for } 1 \leq j \leq t.$$

The computations in the indices of parameters are mod t. We have estimated t elementary contrasts for the direct effects. The previous lemma assures us that all the elementary contrasts for the direct effects are estimable. Let $\hat{T}_{rs}$ be a linear combination of the measurements such that $E(\hat{T}_{rs}) = \tau_r - \tau_s$.

Now consider the entries: $(1,j)$, $(2,j)$, $(1,t+a-b+j)$, $(2,t+a-b+j)$ in $(*)$. We have the following expectations: $E(Y_{1j}) = \mu + \alpha_1 + \beta_j + \tau_{j-1}$, $E(Y_{1,t+a-b+j}) =$ $\mu + \alpha_1 + \beta_{t+a-b+j} + \tau_{a-b+j-1}$, $E(Y_{2j}) = \mu + \alpha_2 + \beta_j +$ $+ \tau_{a+j-1} + \rho_{j-1}$ and $E(Y_{2,t+a-b+j}) = \mu + \alpha_2 + \beta_{t+a-b+j} +$

$+ \tau_{a+j-1} + \rho_{a-b+j-1}$. Then

$E[(Y_{1j}-Y_{2j}+Y_{2,t+a-b+j}-Y_{1,t+a-b+j}) + \hat{T}_{a-b+j-1,j-1}] =$

$= \rho_{a-b+j-1}-\rho_{j-1}$. This holds for $1 \leq j \leq t$. Again, Lemma 4.1 implies that there exists $\hat{R}_{uv}$, a linear combination of measurements, such that $E(\hat{R}_{uv}) = \rho_u - \rho_v$.

When constructing cyclic balanced RM designs it is advisable, for purposes of estimation, that the shifts satisfy the condition $(b-a,t) = 1$, as required in the above proposition.

Corollary 4.1. The designs constructed in Theorem 3.1 are connected for direct and residual effects.

Proof: In these designs (with the exception of $\lambda_1 = 2$ and p odd), the zeros in the first period of the first two pxt subarrays are followed in the order of application by 1 and 2 respectively. Proposition 4.1 assures, therefore, connectedness (for direct and residual effects). In the case of $\lambda_1 = 2$ and p odd, the difference between the same two entries is $(t-2)$. But $(t-2,t) = 1$, since in this case t is odd. So Proposition 4.1 can be successfully applied to this case as well. Connectedness for the residual and direct effects is therefore established.

The previous proposition assures connectedness for direct and residual effects. This means that the ranks of the C-matrices for these effects is at least $t-1$. One of the results that follow will give us a little more insight into what these ranks can be.

Let d be a cyclic balanced $RM(t,p,n,\lambda_1,\lambda_2)$ design with design matrix X. The results we will establish in the remainder of this section hold even when the design d does not satisfy part (a) of Definition 2.1. The restriction $p < t$ present in the previous section won't be assumed here. The $X'X$ matrix of the design d is:

$$X'X = \begin{array}{c|ccccc} & \tau & \rho & \alpha & \beta & \mu \\ \hline \tau & p\lambda_1 I_{t\times t} & \lambda_2 J_{t\times t} - \lambda_2 I_{t\times t} & \lambda_1 J_{t\times p} & P_{11} \; P_{12} \cdots P_{1\lambda_1} & p\lambda_1 1_{t\times 1} \\ \rho & & (p-1)\lambda_1 I_{t\times t} & \lambda_1 J^*_{t\times p} & P_{21} \; P_{22} \cdots P_{2\lambda_1} & (p-1)\lambda_1 1_{t\times 1} \\ \alpha & & & t\lambda_1 I_{p\times p} & J_{p\times n} & t\lambda_1 1_{p\times 1} \\ \beta & & & & pI_{n\times n} & p1_{n\times 1} \\ \mu & & & & & pn_{1\times 1} \end{array}$$

where I is the identity matrix, J is the matrix with all the entries equal to 1, J^* is the matrix with first column entries all zero and the rest of the entries 1. The symbol $1_{a\times 1}$ stays for the column vector of length a with all the entries 1. The incidence matrix between τ and β is $B_1 = \left[P_{11}\ P_{12}\ \cdots\ P_{1\lambda_1}\right]$. Similarly, the incidence matrix between ρ and β is $B_2 = \left[P_{21}\ P_{22}\ \cdots\ P_{2\lambda_1}\right]$. The matrix P_{1j} (respectively P_{2j}) is a txt incidence matrix between τ (respectively ρ) and the units in the j^{th} subarray of the design d. Since each subarray consists of shifts of $(0,1,2,\ldots,t-1)$ it will be enough to find the first row of P_{1j}, the other rows being cyclic rotations of the first. There are exactly p rows in each subarray so the sum of the entries in any row of P_{1j} will be p (all $1 \leq j \leq \lambda_1$). Similarly, the sum in any row of P_{2j} will be p-1 (all $1 \leq j \leq \lambda_1$). If we let P be the following txt matrix:

$$P = \begin{bmatrix} 0 & & & & & & 1 \\ 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & & \\ & & & & 1 & 0 & \\ & & & & & 1 & 0 \end{bmatrix}$$

then $P_{1j} = \sum_{i=0}^{t-1} a_i P^i$ where $P^0 = P^t = I$ and a_i's are nonnegative integers satisfying $\sum_{i=0}^{t-1} a_i = p$. Similarly $P_{2j} = \sum_{i=0}^{t-1} b_i P^i$ where b_i are nonnegative inetgers and $\sum_{i=0}^{t-1} b_i = p-1$. If the design d satisfies condition (a) of Definition 2.1, which is the case with all the designs constructed in Section 3, the integers a_i's and b_i's will all be 1. We are now ready to introduce a couple of useful lemmas.

Lemma 4.2. The joint C-matrix for direct and residual effects is:

$$C(\tau,\rho) = \begin{bmatrix} p\lambda_1 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{1j}P'_{1j} & \lambda_2 J - \lambda_2 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{1j}P'_{2j} \\ \lambda_2 J - \lambda_2 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{2j}P'_{1j} & \lambda_1(p-1)I - \frac{\lambda_1(p-1)}{tp}J - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{2j}P'_{2j} \end{bmatrix}$$

Proof: Partition the X'X matrix as follows:

$$X'X = \begin{bmatrix} X'_1X_1 & X'_1X_2 \\ X'_2X_1 & X'_2X_2 \end{bmatrix}, \text{ where } X'_1X_1 = \begin{bmatrix} p\lambda_1 I_{t\times t} & \lambda_2 J_{t\times t} - \lambda_2 I_{t\times t} \\ & (p-1)\lambda_1 I_{t\times t} \end{bmatrix}$$

$$X'_1X_2 = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1\lambda_1} & p\lambda_1 1_{t\times 1} \\ P_{21} & P_{22} & \cdots & P_{2\lambda_1} & (p-1)\lambda_1 1_{t\times 1} \end{bmatrix} \text{ and}$$

$$X'_2X_2 = \begin{bmatrix} nI_{p\times p} & J_{p\times n} & n1_{p\times 1} \\ & pI_{n\times n} & p1_{n\times 1} \\ & & pn \end{bmatrix}. \text{ The matrix } C(\tau,\rho) \text{ is}$$

obtained as: $C(\tau,\rho) = X'_1X_1 - X'_1X_2(X'_2X_2)^- X'_2X_1$ where

$$(X_2'X_2)^- = \begin{bmatrix} \left(\frac{1}{n}I + \frac{1}{pn}J\right)_{p\times p} & -\frac{1}{pn}J_{p\times n} & 0_{p\times 1} \\ -\frac{1}{pn}J_{n\times p} & \frac{1}{p}I_{n\times n} & 0_{n\times 1} \\ 0_{1\times p} & 0_{1\times n} & 0_{1\times 1} \end{bmatrix}$$

It is a well-known fact that the Moore-Penrose inverse of a symmetric matrix is symmetric, commutes with the matrix and has the same kernel as the matrix. The following lemma gives a way of finding the Moore-Penrose inverse of a polynomial matrix in the $t\times t$ matrix P introduced earlier. Let X be an indeterminate and $f(X)$ a (real or complex) polynomial of degree at most $(t-1)$ in P.

Lemma 4.3. *The Moore-Penrose inverse of* $f(P) = \sum_{i=0}^{t-1} a_i P^i$ *is* $f(P)^- = \sum_{i=0}^{t-1} b_i P^i$, *where* $b_i = c_0 + c_1 w^i + c_2 w^{2i} + \ldots + c_{t-1} w^{(t-1)i}$, w *is a primitive* t^{th} *root of unity and* c_i *equals* $f(w^i)^{-1}$ *if* $f(w^i) \neq 0$ *and* 0 *otherwise*.

Proof: The $t\times t$ permutation matrix P satisfies $P^t = I$ and has in fact $X^t - 1$ as minimal polynomial. Let's introduce the following useful $t\times t$ matrix:

$$S = \frac{1}{\sqrt{t}} \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & w & \cdots & w^j & \cdots & w^{t-1} \\ \vdots & & & \vdots & & \vdots \\ 1 & w^i & \cdots & w^{ij} & \cdots & w^{i(t-1)} \\ \vdots & & & \vdots & & \vdots \\ 1 & w^{t-1} & \cdots & w^{(t-1)j} & \cdots & w^{(t-1)(t-1)} \end{bmatrix}$$

where w is a primitive t^{th} root of unity. The matrix S is unitary and it is easy to show that $S^{-1}PS = D$, where

$$D = \begin{bmatrix} 1 & & & & \\ & w & & & \\ & & \ddots & & \\ & & & w^i & \\ & & & & \ddots \\ & & & & & w^{t-1} \end{bmatrix} .$$

Hence $S^{-1}f(P)S = f(S^{-1}PS) = f(D)$, where

$$f(D) = \begin{bmatrix} f(1) & & & & \\ & f(w) & & & \\ & & \ddots & & \\ & & & f(w^i) & \\ & & & & \ddots \\ & & & & & f(w^{t-1}) \end{bmatrix} .$$

The Moore-Penrose inverse of $f(D)$ is

$$f(D)^- = \begin{bmatrix} c_0 & & & & & \\ & c_1 & & & & \\ & & \ddots & & & \\ & & & c_i & & \\ & & & & \ddots & \\ & & & & & c_{t-1} \end{bmatrix}$$

where c_i equals $f(w^i)^{-1}$ if $f(w^i) \neq 0$ and 0 otherwise. Set $f(P)^- = Sf(D)^- S^{-1}$. It is not difficult to check that $f(P)^-$ defined above is the Moore-Penrose inverse of $f(P)$. The $(i,j)^{th}$ entry in $f(P)^-$ is

$e_{ij} = \frac{1}{t} \sum_{k=0}^{t-1} c_k w^{ki}\, \bar{w}^{kj}$. $f(P)^-$ is a polynomial in P, because

$$te_{ij} = \sum_{k=0}^{t-1} c_k w^{ki}\, \bar{w}^{kj} = \sum_{k=0}^{t-1} c_k w^{ki}\, \bar{w}^{kj} w^k\, \bar{w}^k = \sum_{k=0}^{t-1} c_k w^{k(i+1)} \bar{w}^{k(j+1)} =$$

$= te_{i+1,j+1}$ (the indices in e_{ij} are computed modulo t).
The coefficients b_i can be now defined as
$b_i = \frac{1}{t}(c_0 + c_1 w^i + \ldots + c_{t-1} w^{(t-1)i})$ and hence $f(P)^- = \sum_{i=0}^{t-1} b_i P^i$.

The theorem that follows reveals some properties of the C-matrices for direct and residual effects obtained from cyclic balanced RM designs. For a positive integer d, let $\varphi(d)$ denote the number of positive integers less than d and relatively prime to it.

Theorem 4.1. In a cyclic balanced $RM(t,p,n,\lambda_1,\lambda_2)$ design the C-matrices for direct and residual effects are symmetric polynomials in P. Their ranks equal $(t-1) - \sum_i \varphi(d_i)$ for some (not necessarily same) distinct divisors d_i of t. In these designs contrasts and only contrasts of direct and residual effects may be estimable.

Proof: From $C(\tau,\rho)$ in Lemma 4.2, we obtain at once the two C-matrices:

$$C(\tau) = p\lambda_1 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{1j}P'_{1j} -$$

$$- (\lambda_2 J - \lambda_2 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{1j}P'_{2j})\ f(P)^- \ (\lambda_2 J - \lambda_2 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{2j}P'_{1j})$$

where $f(P) = \lambda_1(p-1)I - \frac{\lambda_1(p-1)}{t\cdot p}J - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{2j}P'_{2j}$, and

$$C(\rho) = \lambda_1(p-1)I - \frac{\lambda_1(p-1)}{t\cdot p}J - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{2j}P'_{2j} -$$

$$- (\lambda_2 J - \lambda_2 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{2j}P'_{1j})\ g(P)^- \ (\lambda_2 J - \lambda_2 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{1j}P'_{2j})$$

where $g(P) = p\lambda_1 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{1j}P'_{1j}$.

Both f and g are symmetric polynomials in P with rational coefficients. In virtue of Lemma 4.3, we can set $C(\tau) = h(P)$ and $C(\rho) = k(P)$, where $h(P)$ and $k(P)$ are themselves symmetric polynomials in P with rational coefficients. The dimension of the kernel of $h(P)$ (respectively $k(P)$) is equal to the number of t^{th} roots of unity which are also roots of $h(X)$ (respectively $k(X)$). Suppose w^i, a d^{th} root of unity for some divisor d of t, is also a root of h. Then all the primitive d^{th} roots of unity are also roots of h as images of w^i under the Galois group of automorphisms of $Q(w^i)$ over Q, the field of rational numbers. There are $\varphi(d)$ primitive d^{th} roots of unity, the roots of the cyclotomic polynomial of w^i over Q. We now show that 1 is always a root of h. As we pointed out previously, P_{2j} is a polynomial in P whose entries along any row add up to $p-1$. Using this fact in calculating $f(1)$, one gets $f(1) = \lambda_1(p-1) - \frac{\lambda_1(p-1)}{tp} t -$ $- \frac{1}{p}\lambda_1(p-1)^2 = \lambda_1(p-1)\left[1 - \frac{1}{p} - \frac{(p-1)}{p}\right] = 0$. The same is true for g, the argument being analogous. Moreover, $h(1) = g(1) - m(1)f(1)^- m'(1) = 0$ since both $g(1)$ and (by Lemma 4.3) $f(1)^-$ are zero. All the remarks about h hold true for k as well. This settles the statement about the ranks of the C-matrices in the theorem. Having always $h(1) = 0$ and $k(1) = 0$ means that the row sums in $C(\tau)$ and $C(\rho)$ are always zero. This implies that the two matrices are singular and hence, of rank at most $t-1$. We know that $t'\tau$ is estimable if and only if t' is in the row span of $C(\tau)$. But t' is in the row span of $C(\tau)$ only if the sum of the entries in t is zero (i.e., only if $t'\tau$ is a linear contrast). The same holds true for the residual effects. This completes the proof.

Let the four $t\times t$ submatrices of $C(\tau,\rho)$ (see Lemma 4.2), which we denote by A_{ij}, be such that

$$C(\tau,\rho) = \begin{bmatrix} A_{11} & A_{12} \\ A'_{12} & A_{22} \end{bmatrix} .$$

Then A_{ij} are polynomials in P. By evaluating A_{11} and A_{22} at the t^{th} roots of unity it can be readily seen that their ranks are always $t-1$. By deleting the last row and last column of these matrices we obtain two $(t-1) \times (t-1)$ matrices which we denote by B_{11} and B_{22}. Then

$$A^-_{11} = \begin{bmatrix} B^{-1}_{11} & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A^-_{22} = \begin{bmatrix} B^{-1}_{22} & 0 \\ 0 & 0 \end{bmatrix}$$

are generalized inverses of A_{11} and A_{22} respectively. The usual row operations in the normal equations lead us to the following result:

Proposition 4.2. The best linear unbiased estimators in a cyclic balanced $RM(t,p,n,\lambda_1,\lambda_2)$ design for the estimable linear contrasts $t'\tau$ and $t'\rho$ are respectively:

$$t'\hat{\tau} = t'C(\tau)^{-}\left[\left(X_{11}'-X_{11}'X_2(X_2'X_2)^{-}X_2'\right)- A_{12}A_{22}^{-}\left(X_{12}'-X_{12}'X_2(X_2'X_2)^{-}X_2'\right)\right]Y$$

and

$$t'\hat{\rho} = t'C(\rho)^{-}\left[\left(X_{12}'-X_{12}'X_{11}(X_2'X_2)^{-}X_2'\right)-A_{21}A_{11}^{-}\left(X_{11}'-X_{11}'X_2(X_2'X_2)^{-}X_2'\right)\right]Y$$

where P_{1j} = incidence matrix between direct effects and units in the j^{th} subarray.

P_{2j} = incidence matrix between residual effects and units in the j^{th} subarray.

$$A_{11} = p\lambda_1 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{1j}P_{1j}'$$

$$A_{12} = \lambda_2 J - \lambda_2 I - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{1j}P_{2j}'$$

$$A_{22} = \lambda_1(p-1) - \frac{\lambda_1(p-1)}{t\cdot p}J - \frac{1}{p}\sum_{j=1}^{\lambda_1} P_{2j}P_{2j}'$$

$$C(\tau) = A_{11} - A_{12}A_{22}^{-}A_{12}'$$

$$C(\rho) = A_{22} - A_{12}'A_{11}^{-}A_{12}$$

$$(X_2'X_2)^- = \begin{bmatrix} \left(\frac{1}{n}I + \frac{1}{pn}J\right)_{p\times p} & -\frac{1}{pn}J_{p\times n} & 0_{p\times 1} \\ -\frac{1}{pn}J_{n\times p} & \frac{1}{p}I_{n\times n} & 0_{n\times 1} \\ 0_{1\times p} & 0_{1\times n} & 0_{1\times 1} \end{bmatrix}$$

X_{11} = columns in the design matrix corresponding to the direct effects.

X_{12} = columns in the design matrix corresponding to the residual effects.

X_2 = columns in the design matrix not in X_{11} or X_{12}.

Y = vector of observations (ordered the same way as in the design matrix).

Remark. When the design is connected $C(\tau)^-$ and $C(\rho)^-$ can be computed the same way as A_{11}^-. Otherwise Lemma 4.3 can be used.

We have $C(\tau)\hat{\tau} = TY$ and $C(\rho)\hat{\rho} = RY$ where T and R have been explicitly given above. It can be shown, in a straightforward manner, that both T and R can be partitioned into $\lambda_1 p$ submatrices of size $t\times t$ each and each such submatrix is a polynomial in the matrix P, with zero row sums. Let these partitions be $T = [T_1 \ldots T_{\lambda_1 p}]$ and $R = [R_1 \ldots R_{\lambda_1 p}]$.

__Theorem 4.2__. __In a cyclic balanced and connected__ $RM(t,p,n,\lambda_1,\lambda_2)$ __design if the__ b.l.u.e. __of__ $\tau_0 - \tau_1$ (__respectively__ $\rho_0 - \rho_1$) __is known, then the__ b.l.u.e. __of any linear contrast of direct__ (__respectively residual__) __effects can be derived from it__.

__Proof:__ Recall that for a given vector $\ell' = (\ell_1,\ell_2,\ldots,\ell_t)$ its shift starting at i is the vector $(\ldots \ell_t,\ell_1,\ell_2 \ldots)'$, denoted by $i(\ell')$, in which ℓ_1 is in the i^{th} position. Indices are computed modulo t. Suppose $\ell' C(\tau)\hat{\tau} = \hat{\tau}_0 - \hat{\tau}_1$ where $\ell' = (\ell_1,\ell_2,\ldots,\ell_t)$. Then the b.l.u.e. of $\tau_0 - \tau_1$ is $(\ell' T_1 \ldots \ell' T_{\lambda_1 p})Y$. Because all the matrices involved are polynomials in P, $\hat{\tau}_i - \hat{\tau}_{i+1} = i(\ell')C(\tau)\hat{\tau}$. Hence the b.l.u.e. of $\tau_i - \tau_{i+1}$ is $(i(\ell')T_1 \ldots i(\ell')T_{\lambda_1 p})Y$ for all $0 \leq i \leq t-1$. This can be formulated in terms of the entries of the design as follows: The first row of treatments in each subarray of size $p\times t$ should be always the same, say $0,1,\ldots,t-1$. Associate the weights (coefficients) of the b.l.u.e. of $\tau_0 - \tau_1$ with the corresponding entries (cells) in this design. To compute the b.l.u.e. of $\tau_i - \tau_{i+1}$, leave the weights associated with the same cells, but do cyclic column rotations in each subarray until the cells which were filled by 0 will be filled by i. The weight associated with a certain cell will then become the coefficient of the observation which falls in that cell after these cyclic rotations of columns. An example will follow. Any other linear contrast of direct effects can be expressed as a linear combination of the t elementary contrasts considered above. The b.l.u.e. of the linear contrast is then the same linear combination of the b.l.u.e.'s of these elementary contrasts. The same holds true for residual effects.

Let's illustrate by a small example what the previous result says. Given the RM(3,2,6,2,1)

$$\begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{matrix}$$

the b.l.u.e. for $\tau_0 - \tau_1$ is

$$\tfrac{1}{3}\Big(Y_{11} + Y_{12} - 2Y_{13} - Y_{14} - Y_{15} + 2Y_{16} - $$
$$- Y_{21} - Y_{22} + 2Y_{23} + Y_{24} + Y_{25} - 2Y_{26}\Big) .$$

The b.l.u.e. of $\tau_2 - \tau_0$ can be found by writing the design as

$$\begin{matrix} 2 & 0 & 1 \\ 0 & 1 & 2 \end{matrix} \qquad \begin{matrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{matrix}$$

and using the same (unchanged) weights as previously:

$$\tau_2 - \tau_0 = \tfrac{1}{3}\big(Y_{13} + Y_{11} - 2Y_{12} - Y_{16} - Y_{14} + 2Y_{15} - \\ - Y_{23} - Y_{21} + 2Y_{22} + Y_{26} + Y_{24} - 2Y_{25}\big)$$

The b.l.u.e. of $2\tau_0 - \tau_1 - \tau_2$ can be found by subtracting the b.l.u.e. of $\tau_2 - \tau_0$ from that of $\tau_0 - \tau_1$.

A list of partitions for values of t between 3 and 20 is attached. From each partition one can exhibit the corresponding balanced connected designs as shown in the proof of Proposition 3.2 and the two examples following it. A comprehensive bibliography on Repeated Measurements Designs, prior to 1975, can be found in A. Hedayat and K. Afsarinejad (1975).

Appendix

Proof of Theorem 3.1.

We proceed in showing that the conditions stated in Proposition 3.2 are satisfied for all the partitions listed after the statement of Theorem 3.1. In all these cases it is straightforward to check that

$\bigcup_{i=1}^{\lambda_1} P_i = \{L,2,3,\ldots,t-1\}$ (disjoint) and that $|P_i| = p-1$ for all $1 \leq i \leq \lambda_1$.

case 1. (p even). To check the condition $\sum_{\ell=j}^{k} s_{i\ell} \not\equiv 0 \pmod t$ for all $1 \leq j < k \leq (p-1)$, where $P_i = \{(s_{ij})\}_{1 \leq j \leq (p-1)}$ is the P_i defined above, we note that $s_{ij} + s_{i,j+1} = (t-1)$ for all odd $1 \leq j \leq (p-1)$ and all $1 \leq i < \lambda_1$. This fact enables us to classify an arbitrary sum $\sum_{\ell=j}^{k} s_{i\ell}$, as above, in one of the following four patterns:

(a) $[(i+j\lambda_1) + (t-1) - (i+j\lambda_1)]+\ldots+[(i+k\lambda_1) + (t-1) - (i+k\lambda_1)$ for all $1 \leq j < k \leq \frac{(p-4)}{2}$. If for some j and k this sum is equal to mt (for some $m \in Z$) we have $(k-j+1)(t-1) = mt$. Or, $(k-j+1) = (k-j+1-m)t$. But $2 \leq (k-j+1) \leq \frac{(p-1)}{2} < t$, which leads to a contradiction.

(b) $[(i+j\lambda_1) + (t-1) - (i+j\lambda_1)]+\dots+[(i+k\lambda_1) + (t-1) - (i+k\lambda_1)]$ $+ (i+(k+1)\lambda_1)$ for all $1 \leq j < k \leq \frac{(p-4)}{2}$. If for some j and k this sum is equal to mt, we get $(-k+j-1+i+(k+1)\lambda_1) = (m-(k-j+1))t$. But $1 \leq i+j =$ $-k + j-1 + i + (k+1) \leq (-k+j-1+i+(k+1)\lambda_1) =$ $= (k+1)\lambda_1 + i - 1 - (k-j) < \frac{(p-2)}{2}\lambda_1 + \lambda_1 = \frac{p}{2}\lambda_1 \leq (p-1)\lambda_1 =$ $(t-1) < t$. This contradicts the fact that $(-k+j-1+i+(k+1)\lambda_1$ is a multiple of t.

(c) $(t-1) - (i+(j-1)\lambda_1) + [(i+j\lambda_1) + (t-1) - (i+j\lambda_1)] +$ $\dots+ [(i+k\lambda_1) + (t-1) - (i+k\lambda_1)]$ for all $0 \leq j-1 < k \leq \frac{(p-4)}{2}$. This sum is a multiple of t only if $(i+(j-1)\lambda_1+k-j+2)$ is a multiple of t. But $(i+(j-1)\lambda_1+k-j+2)$ is not a multiple of t since $2 \leq (i+(j-1)\lambda_1+k-j+2) < \lambda_1 + \frac{(p-4)}{2}\lambda_1 + \frac{(p-4)}{2} + 2 <$ $< \frac{(p-1)}{2}\lambda_1 + \frac{(p-1)}{2} < \frac{(t-1)}{2} + \frac{(t-1)}{2} = (t-1) < t$.

(d) $(t-1) - (i+(j-1)\lambda_1) + [i+j\lambda_1+(t-1)-(i+j\lambda_1)] +$ $\dots+ [(i+k\lambda_1+(t-1)-(i+k\lambda_1)] + i + (k+1)\lambda_1$ for all $0 \leq (j-1) < (k+1) \leq \frac{(p-2)}{2}$. Proceeding as previously, this sum is a multiple of t for some j and k, only if $(k-j+2)(\lambda_1-1)$ is a multiple of t. But $2 \leq (k-j+2)(\lambda_1-1) < (k+1)\lambda_1 < \frac{(p-1)}{2}\lambda_1 = \frac{(t-1)}{2} < t$.

We showed that, for $1 \leq i < \lambda_1$, the conditions on P_i's required in Proposition 3.2 are satisfied. Since $i < \lambda_1$ has not been used in any of the previous inequalities, the previous four patterns apply also to P_{λ_1} with the last entry removed. So we only have to consider the cases that involve the last entry of P_{λ_1}: i.e., t-1.

(a_1) $\ell(t-1) + (t-1)$, for $0 \leq \ell \leq \frac{(p-2)}{2}$, where ℓ is the number of successive pairs of entries in P_{λ_1} that add up to $(t-1)$. Clearly $\ell(t-1) + (t-1)$ is a multiple of t only if $(\ell+1)$ is a multiple of t. But $1 \leq \ell+1 \leq \frac{(p-2)}{2} < t$.

(b_1) $(t-1) - j\lambda_1 + \ell(t-1) + (t-1)$, for $0 \leq \ell \leq \frac{(p-4)}{2}$ and $(j+\ell) = \frac{(p-2)}{2}$. Here ℓ has the same meaning as before. This sum is a multiple of t only if $(\ell+2+j\lambda_1)$ is also a multiple of t. Since $2 \leq (\ell+2+j\lambda_1) \leq (\ell+2) + \left(\frac{(p-2)}{2} - \ell\right)\lambda_1 \leq \frac{p}{2} + \frac{(p-2)}{2}\lambda_1 \leq \frac{p}{2} + \frac{(p-1)}{2}\lambda_1 \leq \frac{(p+t-1)}{2} \leq \frac{2t-1}{2} < t$, the proof for p even is completed.

<u>case</u> <u>2.1</u>. (p odd and $\lambda_1 > 2$). The same four patterns arise here as in the case of p even and they are ruled out the same way. Therefore, we only have to consider the sums that involve the last entry of P_{λ_1}.

In what follows, let ℓ be the number of successive pairs of entries in P_{λ_1} that add up to $(t-1)$.

(a_2) $\ell(t-1) + \frac{(p-1)}{2}\lambda_1 + (t-1)$; for all $0 \leq \ell \leq \frac{(p-3)}{2}$.

Clearly, $\ell(t-1) + \frac{(p-1)}{2}\lambda_1 + (t-1) = mt$ (for some $m \in Z$) only if $(-\ell + \frac{(p-1)}{2}\lambda_1 - 1) = (m-\ell-1)t$. But $0 < \frac{(p-1)}{2}(\lambda_1-1) \leq (-\ell + \frac{(p-1)}{2}\lambda_1-1) \leq \frac{(p-1)}{2}\lambda_1 = \frac{(t-1)}{2} < t$.

(b_2) $(t-1) - j\lambda_1 + \ell(t-1) + \frac{(p-1)}{2}\lambda_1 + (t-1)$; for all $0 \leq \ell \leq \frac{(p-3)}{2}$, and $j = \frac{(p-3)}{2} - \ell$. If this sum is mt (for some $m \in Z$), then $\left(\frac{(p-1)}{2} - j\right)\lambda_1 - \ell-2 = (m-\ell-2)t$. Since $-t < -\ell \leq \left(\frac{(p-1)}{2} - j\right)\lambda_1 - \ell - 2 \leq \frac{(p-1)}{2}\lambda_1 = \frac{(t-1)}{2} < t$, we can only have $\left(\frac{(p-1)}{2} - j\right)\lambda_1 = \ell - 2$. Substituting for $j = \frac{(p-3)}{2} - \ell$, we get $\lambda_1 = \frac{\ell+2}{\ell+1}$. But $\frac{\ell+2}{\ell+1}$ is a positive integer if and only if $\ell = 0$, in which case $\lambda_1 = 2$.

<u>case 2.2</u>. (p odd and $\lambda_1 = 2$). P_1 is defined the same way as in the previous case, so we should only examine the partial sums in P_2. But the partial sums in P_2 with the first entry removed are satisfactory by case 1. Cases involving the first entry in P_2 should only be considered. There are two possibilities: (a_3) $(t-1) + \ell(t-1)$ and (b_3) $(t-1) + \ell(t-1) + 2j$, where $0 \leq \ell \leq \frac{(p-3)}{2}$ and (in (b_3)) $j = \ell + 1$. In both these cases the remainder modulo t is $\ell + 1$ and $1 \leq \ell + 1 \leq \frac{p-1}{2} < t$. This concludes the construction of the designs and the proof of the theorem.

t arbitrary, p=2: |(1)|(2)|(3)| ... |(t-1)|

t=5, p=3: |(1,3)|(4,2)|

t=7, p=3: |(1,5)|(2,4)|(3,6)|

t=7, p=4: |(1,5,3)|(2,4,6)|

t=9, p=3: |(1,7)|(2,6)|(3,5)|(4,8)|

t=9, p=5: |(1,7,3,5)|(8,2,6,4)|

t=10, p=4: |(1,8,4)|(2,7,5)|(3,6,9)|

t=11, p=3: |(1,9)|(2,8)|(3,7)|(4,6)|(5,10)|

t=11, p=6: |(1,9,3,7,5)|(2,8,4,6,10)|

t=13, p=3: |(1,11)|(2,10)|(3,9)|(4,8)|(5,7)|(6,12)|

t=13, p=4: |(1,11,5)|(2,10,6)|(3,9,7)|(4,8,12)|

t=13, p=7: |(1,11,3,9,5,7)|(12,2,10,4,8,6)|

t=15, p=3: |(1,13)|(2,12)|(3,11)|(4,10)|(5,9)|6,8)|(7,14)|

t=15, p=8: |(1,13,3,11,5,9,7)|(2,12,4,10,6,8,14)|

t=16, p=4: |(1,14,6)|(2,13,7)|(3,12,8)|(4,11,9)|(5,10,15)|

t=16, p=6: |(1,14,4,11,7)|(2,13,5,10,8)|(3,12,6,9,15)|

t=17, p=3: |(1,15)|(2,14)|(3,13)|(4,12)|(5,11)|(6,10)|(7,9)|(8,16)|

t=17, p=5: |(1,15,5,11)|(2,14,6,10)|(3,13,7,9)|(4,12,8,16)|

t=17, p=9: |(1,15,3,13,5,11,7,9)|(16,2,14,4,12,6,10,8)|

t=19, p=3: |(1,17)|(2,16)|(3,15)|(4,14)|(5,13)|(6,12)|(7,11)|
|(8,10)|(9,18)|

t=19, p=4: |(1,17,7)|(2,16,8)|(3,15,9)|(4,14,10)|(5,3,11)|(6,12,18)|

t=19, p=7: |(1,17,4,14,7,11)|(2,16,5,13,8,10)|(3,15,6,12,9,18)|

t=19, p=10: |(1,17,3,15,5,13,7,11,9)|(2,16,4,14,6,12,8,10,18)|

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